oncommutative spheres and the AdS/CFT corresponden

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We present a further argument in favor of non-commutativity of spheres in the AdS/CFT correspondence. Concentrating on the SCFT of S_N orbifold we use results for three point correlators (at finite N). Comparison with dual calculations on a non-commutative sphere is given with the characteristic feature of non-commutativity clearly identified.

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1. Introduction

The AdS/CFT correspondence [1], [2], [3] provides a constructive approach to supergravity, and closed string theory in curved backgrounds. Its basis is the large N expansion of Yang-Mills type theory which turns into a loop expansion of gravity. The existence of a deductive procedure has prompted questions concerning any modification that the emerging gravity could exhibit. In particular a proposal was made in [4], [5], that the curved SUGRA background $AdS \times S$ is non-commutative with the noncommutativity parameter given by $\frac{1}{N}$. This follows an earlier proposal for q-deformation given in a framework of a 1d matrix model [6]. The extension to higher cases was further discussed in [7]. The non-commutativity of $AdS \times S$ space naturally incorporates the exclusion principle of [8] and stands to have important implications on the physics of black holes.

It also conforms to a general principle of [9] that string/M-theory inherently contains a space-time uncertainty(see [10]). Recently, a physical argument for the noncommutativity and the associated cutoff was given by [11] based on study of brane motions on spheres. This discussion involves a mechanism [12], [13] by which gravitons are polarized into extended membranes which lead to non-commutativity [14]. Other examples are given in [15],[16] .

It is clearly important to further clarify the nature of non-commutativity in AdS spaces. In comparison with the noncommutativity directly induced by an external B-field for example the origin of noncommutativity in AdS is more nontrivial to exhibit, it involves the parameter $\frac{1}{N}$ and is consequently nonperturbative in nature.

In this paper, we discuss further evidence for the above noncommutativity in AdS/CFT. The discussion is based on the S_N orbifold model that already served as the basis for arguments presented in [4], [5]. In this model, one is able to perform explicit calculations of three point interactions at finite N and study their behavior. In this way, we find an explicit signature for non-commutativity of the corresponding sphere.

The content of the paper is as follows. In section 2, we summarize the finite N results for three-point correlations in the orbifold CFT. After reviewing the (super)gravity calculation in commutative space-time in section 3 we evaluate the modifications due to a non-commutative sphere exhibiting agreement with the CFT formula.

2. Results from S_N orbifold

In this section, we review the results of CFT dual to the gravity in the case of $AdS_3 \times S^3$ obtained in [5]. We also use the extension of these results to the nonextremal case that can be read off from recent work of [17].

The SCFT in question is defined on symmetric product $S^N(M)$, where M is either T^4 or $K3$. The field content of the theory consists of: $4N$ real free bosons $X_I^{a\dot{a}}$ representing the coordinates of the torus for example and their superpartners $4N$ the fermions $\Psi_I^{\alpha\dot{a}}$, where $I = 1,..,N$, $\alpha, \dot{\alpha} = \pm$ are the spinorial S^3 indices, and $a, \dot{a} = 1, 2$ are the spinorial indices on T^4. In essence, the field content of the theory is determined to be $4N$ real free bosons and $2N$ Dirac free fermions, giving a central charge $c = 6N$. One has left and right superconformal symmetry with the corresponding currents. The lowest modes of this currents $\{L_{0,\pm 1}, G^{\alpha a}_{\pm\frac{1}{2}}, J_0^{\alpha\beta}\}$ together with their right counterparts generate the $SU(2|1,1)_L \times SU(2|1,1)_R$ symmetry which in the AdS/CFT correspondence translates into the superisometries of the $AdS_3 \times S^3$. In addition, it is possible to construct other symmetries commuting with the previous set and related to global T^4 rotations. Even though the underlying CFT on T^4 is free, the complexity is given by the non-trivial implementation of the S_N symmetry of the orbifold. Invariant chiral primary operators were constructed in [4], [5]. A basic role is played by the twist operators that impose the twisted the boundary conditions :

$$X_I(z\, e^{2\pi i}, \bar{z}\, e^{-2\pi i}) = X_{I+1}(z, \bar{z}),\; I = 1..n-1,\; X_n(z\, e^{2\pi i}, \bar{z}\, e^{-2\pi i}) = X_1(z, \bar{z}) \tag{2.1}$$

One first has the Z_n twist operators that can be used to build S_N invariant chiral primaries by averaging over S_N. The twist operators play a distinguished role in the chiral ring in the sense that they can be used to generate the rest using the ring structure. In the correspondence with gravity on $ADS_3 \times S^3$ one achieves a one-one correspondence with single particle states. The computation of two and three point functions (for extremal momenta) was given in [5] and has recently been extended [17].

We present the three-point function for chiral primaries $O_n^{(0,0)}$, where n is related with the length of the cycle used to construct the operator. The index n is also related with l the angular momentum on S^3 in gravity as $n = l + 1$ (we remind that the isometry group for S^3 is $SO(4) = SU(2) \times SU(2)$ and that l is an angular momentum in the diagonal

$SU(2)$). The correlation functions in terms of $n = l_1 + 1$, $k = l_2 + 1$ and $n + k - 1 = l_3 + 1$ ($l_3 = l_1 + l_2$ case) are:

$$\langle O^{(0,0)\,\dagger}_{n+k-1}(\infty)\, O^{(0,0)}_{k}(1)\, O^{(0,0)}_{n}(0)\rangle = \left(\frac{(N-n)!\,(N-k)!\,(n+k-1)^3}{(N-(n+k-1))!\,N!\,n\,k}\right)^{\frac{1}{2}} \tag{2.2}$$

The expressions for correlation functions contain two types of terms, one that does not has N dependence and the one with N dependence that will correspond in some gravity picture to some noncommutative deformation and cutoff. We would like to extend the result beyond the case $l_1 + l_2 = l_3$ and we use the results obtained in [17] for a simpler orbifold theory. Then, in term of angular momentum $l_{1,2,3}$, the factor coming from the noncommutative deformation is:

$$\frac{((N-1-l_1)!(N-1-l_2)!(N-1-l_3)!)^{\frac{1}{2}}}{(N-1-\frac{l_1+l_2+l_3}{2})!} \tag{2.3}$$

It is this factor that we would like to recover in gravity as it represents the finite N effects deformations coming from the dual CFT. For that reason we will study in the next sections a scalar field theory on fuzzy sphere (N) where the $N \to \infty$ case reproduces the sphere case.

In our previous work we have argued for non-commutative (q-deformation of space-time)in the AdS/CFT correspondence.In the next section we will give some general arguments in favour of a connection between computation of correlation functions on q-$ADS2 \times S^2$ and q-$ADS3 \times S^3$. The q-sphere can be defined as quotient of $SU(2)_q$ [18] and belongs to the classification of [19]. There are also transformations between q-spheres with manifest $SU(2)_q$ symmetry and spheres with manifest $SU(2)$ symmetry. For generic q this takes the form of a connection between the classical sphere and the q-sphere. For roots of unity this takes the form of a connection between the q-sphere and the fuzzy sphere. The technical reason for these connections is essentially the deformation maps between U_q generators and the generators of the classical symmetry discussed in [20]. Applying these transformations to both the algebra of functions on the sphere and to the symmetry generators acting on the algebra gives a transformation between q-sphere with U_q suymmetry and classical sphere with $U(SU(2))$ symmetry for generic q. This can be expected to give lead at roots of unity to a transformation between fuzzy sphere and q-sphere, for $q = e^{\frac{i\pi}{2N}}$ and the fuzzy sphere corresponding to $N \times N$ matrices. The connection between q and N can be guessed simply by matching the spectrum of unitary reps of the U_q symmetry with the spectrum of spins appearing in the adjoint action of $SU(2)$ on the fuzzy sphere. This is a non-commutative version of a diffeomorphism which should be a symmetry in these applications of quantum spheres to non-commutative gravity.

3. Field theory on $AdS_3 \times S^3$ and $AdS_2 \times S^2$

In evaluating of correlation functions on the SUGRA side one has the following two-step process. The AdS dependence is projected to the boundary of the AdS space-time using the bulk to boundary propagator. For the sphere one expands in terms of spherical harmonics:

$$\phi = \sum_I \phi_I \, Y^I. \tag{3.1}$$

The signature of noncommutative space that we are going to exhibit is associated with the S (sphere) part of this calculation. The AdS part does not lead to such a characteristic behavior and in much of what will be presented can be ignored. Nevertheless, since in the evaluation of three point interactions (in the commutative case) there are major cancellations between the two one should pay attention to the full calculation. We will first discuss the calculation in the commuting case summarizing notation contained in previous works. We can discuss for simplicity a massless dimensional scalar field theory with cubic interaction:

$$S = \int dx \sqrt{g}((\partial\Phi)^2 + \lambda\Phi^3 + \ldots) \tag{3.2}$$

where the integral is over the $AdS \times S$ space, g is the corresponding metric and λ represents the coupling constant for the cubic interaction.

In studying field theory on products $AdS \times S$ space, the wavefunctions factorize into AdS and S components

$$\Phi = \sum \Phi(\rho, t, \phi)\Psi_I(\vec{n})$$

where $\Psi_I(\vec{n})$ denote the spherical functions on the sphere S. For the case S_3, one has the well known $D^{\ell}_{mm'}(\theta, \varphi, \psi)$ functions. For three point functions, one has schematically

$$\langle \Phi\Phi\Phi \rangle_{AdS} \, \langle \Psi_{\ell_1} \Psi_{\ell_2} \Psi_{\ell_3} \rangle_S$$

The sphere contributions (on which we will concentrate) exhibit typically the Clebsch-Gordon coefficients

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1' & m_2' & m_3' \end{pmatrix} F\left(\ell_1, \ell_2, \ell_3; N\right)$$

and a factor $F(\ell; N)$ solely dependent on the magnitudes of angular momenta and N. In the non-commutative case, this is the structure of a star product

$$Y^{\ell_1} * Y^{\ell_2} = \sum \left(c\ell - G\right) f\left(\ell_1 \ell_2 \ell_3; N\right) Y^{\ell_3}$$

For studying the form of the "fusion coefficient", $F(\ell_1\ell_2,\ell_3:N)$, it is sufficient to consider the reduction of the sphere S_3 to S_2. One knows that in particular

$$D^{\ell}_{mm'=0} = Y_{\ell m}(\theta,\phi)$$

i.e. we have spherical harmonics on S_2. The nature of non-commutativity is essentially the same for the $AdS_3 \times S_3$ space or $AdS_2 \times S_2$ which is its $U(1)\times U(1)$ coset. So in what follows for simplicity of the calculation, we will discuss the latter.

We use the following representation for spherical harmonics:

$$Y^I = \Omega^I_{i_1\dots i_l}\frac{x^i_1\dots x^{i_l}}{\rho^l}, \tag{3.3}$$

where Ω^I is a traceless, symmetric, l-index tensor with indices $i_k = 1\dots 3$, x^i are the coordinates of the three dimensional flat space and $\rho = \sqrt{(x^1)^2+(x^2)^2+(x^3)^2}$. Y^I is an eigenvector of the sphere laplacian with eigenvalue $l(l+1)$. We also assume that the tensors Ω^I are normalized in the sense: $\Omega^I_{i_1\dots i_l}\Omega^J_{i_1\dots i_l} = \delta^{IJ}$ for equal index number and 0 otherwise. By straightforward computations presented in the appendix 1 we obtain the following expressions for the product of two harmonics ((I,l_1), (J,l_2) indices) integrated over the sphere:

$$\langle Y^I Y^J\rangle \equiv \frac{1}{4\pi}\int_{S^2} Y^I Y^J = \frac{\pi^{\frac{1}{2}}\Gamma(l+1)}{2^{l+1}\Gamma(l+\frac{3}{2})}\delta^{IJ}, \tag{3.4}$$

For the product of three harmonics ((I,l_1), (J,l_2), (K,l_3) indices) we obtain:

$$\langle Y^I Y^J Y^K\rangle \equiv \frac{1}{4\pi}\int_{S^2} Y^I Y^J Y^K = \frac{\pi^{\frac{1}{2}}\Gamma(l_1+1)\Gamma(l_2+1)\Gamma(l_3+1)}{2^{\frac{\Sigma}{2}+1}\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(\alpha_3+1)\Gamma(\frac{\Sigma+3}{2})}C^{IJK} \tag{3.5}$$

where $\Sigma = l_1+l_2+l_3$, $\alpha_1 = \frac{1}{2}(-l_1+l_2+l_3)$, $\alpha_2 = \frac{1}{2}(l_1-l_2+l_3)$, $\alpha_3 = \frac{1}{2}(l_1+l_2-l_3)$ and:

$$C^{IJK} = \Omega^I_{i_1\dots i_{\alpha_3}k_1\dots k_{\alpha_2}}\Omega^J_{i_1\dots i_{\alpha_3}j_1\dots j_{\alpha_1}}\Omega^J_{j_1\dots j_{\alpha_1}k_1\dots k_{\alpha_2}} \tag{3.6}$$

We multiply the spherical harmonics with appropriate factors in order to normalize them. After we integrate over the sphere we obtain the following AdS_2 action:

$$S = \int_{AdS_2} d^2x\sqrt{g}(\partial\phi_I\partial\phi_I + l(l+1)\phi_I\phi_I + \lambda A^{IJK}\phi_I\phi_J\phi_K) \tag{3.7}$$

where g is now the AdS_2 metric and:

$$A^{IJK} = \frac{\pi^{-\frac{1}{4}}2^{\frac{1}{2}}\left(\Gamma(l_1+1)\Gamma(l_1+1)\Gamma(l_3+1)\Gamma(l_1+\frac{3}{2})\Gamma(l_2+\frac{3}{2})\Gamma(l_3+\frac{3}{2})\right)^{\frac{1}{2}}}{\Gamma(\frac{\Sigma+3}{2})\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(\alpha_3+1)}. \tag{3.8}$$

The action in AdS_2 can be related to the boundary action using the procedure and expressions developed in [3], [21] (see also [22]). The two- and three- point expressions for the constants in the correlation functions of the boundary operators O_I (corresponding to ϕ_I) are:

$$\begin{aligned}
\langle O_I O_J \rangle &= \delta_{IJ} \frac{\Gamma(\Delta_I + 1)}{\pi^{\frac{1}{2}} \Gamma(\Delta_I - \frac{1}{2})}, \\
\langle O_I O_J O_K \rangle &= -\frac{\lambda A^{IJK}}{2\pi} \frac{\Gamma(\frac{-\Delta_I+\Delta_J+\Delta_K}{2})\Gamma(\frac{\Delta_I-\Delta_J+\Delta_K}{2})\Gamma(\frac{\Delta_I+\Delta_J-\Delta_K}{2})\Gamma(\frac{\Delta_I+\Delta_J+\Delta_K-2}{2})}{\Gamma(\Delta_I - \frac{1}{2})\Gamma(\Delta_J - \frac{1}{2})\Gamma(\Delta_K - \frac{1}{2})},
\end{aligned} \tag{3.9}$$

where Δ for each operator is $l+1$. We redefine the operators such that the constant in the two-point function is δ_{IJ}. After introducing all the factors, those coming from normalization and A^{IJK}, we obtain the following expression for the three-point correlation functions:

$$\begin{aligned}
\langle O_I O_J O_K \rangle = &-\frac{\lambda C^{IJK}}{(2\pi)^{\frac{1}{2}}} \left(\frac{(l_1+\frac{1}{2})(l_2+\frac{1}{2})(l_3+\frac{1}{2})}{(l_1+1)(l_2+1)(l_3+1)} \right)^{\frac{1}{2}} \frac{\Gamma(\alpha_1+\frac{1}{2})\Gamma(\alpha_2+\frac{1}{2})\Gamma(\alpha_3+\frac{1}{2})}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(\alpha_3+1)} \times \\
&\times \frac{\Gamma(\frac{\Sigma-1}{2})}{\Gamma(\frac{\Sigma+3}{2})}.
\end{aligned} \tag{3.10}$$

We observe that in our case the scalar has a mass given by the sphere laplacian $\sqrt{l(l+1)}$. In the case of $AdS_2 \times S_2$ gravity, much of the analysis is in terms of chiral primary fields. The chiral primary fields are combinations of fields coming from four dimensional gravity that have the lowest possible AdS_2 mass for a given l: $\sqrt{l(l-1)}$. We can assume that we do not start with such a simple theory as in (3.2), but with one that after sphere reduction leads to the lowest mass, appropriate for a chiral primary field [3]. For such a field, the corresponding Δ is l. In such a theory, we also consider a simple cubic interaction and obtain a qualitative picture for the sphere reduction. In this case, the three-point correlation functions for chiral primaries operators obtained in the end are simpler:

$$\langle O_I O_J O_K \rangle = -\frac{\lambda C^{IJK}}{(2\pi)^{\frac{1}{2}}} \frac{4\left((l_1^2-\frac{1}{4})(l_2^2-\frac{1}{4})(l_3^2-\frac{1}{4})\right)^{\frac{1}{2}}}{\alpha_1\alpha_2\alpha_3(\Sigma^2-1)}. \tag{3.11}$$

We observe that the correlation functions in the case of chiral primary (3.11) is much simpler than (3.10). This is one of the features observed in all gravity cases of $AdS_p \times S^p$ reduction for chiral primary operators. The cancellation of factors appears between those coming from sphere and those coming from AdS spaces. We also note that (3.11) gives a divergent result for extremal cases like $l_1 + l_2 = l_3$. The divergences are due to our simplified model and they disappear in a realistic model. In the case of gravity, both the quadratic and cubic terms contain higher derivatives of the fields and these are responsible for both lower mass and consistent three-point correlation functions.

4. Fuzzy sphere and spherical harmonics

In this section, we review the definition and the properties of the fuzzy sphere. We also give, the definition for spherical harmonics, for the integration over the fuzzy sphere and present a consistent method to compute (following the discussion for the commutative case) the relevant quantities such as the normalization constants (3.4) and the three harmonic interaction (3.5).

The definition of the fuzzy sphere (for a review see [23]) is given in terms of an algebra of polynomials in X^i, $i = 1 \ldots 3$ subject to the following constraints:

$$\begin{aligned} &[X^i, X^j] = i\epsilon_{ijk} X^k, \\ &(X^1)^2 + \ldots + (X^3)^2 = \rho^2, \end{aligned} \tag{4.1}$$

where ρ^2 is a constant equal to $\frac{N^2-1}{4}$ (N is a positive integer measuring the fuzziness of the sphere). From the above expression, we notice that such a deformation preserves the $SO(3)$ symmetry of the sphere. We can represent the X's (and the algebra) as hermitian operators in the $SO(3)$ representation having spin $\frac{N-1}{2}$. As such, the coordinates are now hermitian $N \times N$ matrices. In such a representation we define the operation replacing the integral over the sphere as:

$$\frac{1}{4\pi} \int_{S^2} (\ldots) \to \frac{1}{N} Tr(\ldots) \tag{4.2}$$

where Tr is the trace in the $\frac{N-1}{2}$ representation and $\ldots$ mean a function on the sphere. It is also straightforward to represent the $su(2)$ symmetry (generators J_i) in this algebra:

$$J_i f(X) = [X^i, f(X)], \qquad i = 1 \ldots 3 \tag{4.3}$$

The spherical harmonics are constructed in the same way as in commutative sphere (3.3) replacing the commutative coordinates x^i with the noncommutative ones X^i. It is straightforward to prove that the vector space of symmetric traceless polynomial of degree l is left invariant by the $su(2)$ generators and that it forms an irreducible representation with highest weight l.

The symmetric polynomials in X of any degree appear in the Taylor series of $exp(iJX) = exp(iJ_iX^i)$, where J's are regular commutative numbers. In order to compute the normalization constants and the three point interaction we construct the following quantity:

$$I(J^1, J^2, J^3) = \frac{1}{N}Tr(e^{iJ^1X}e^{iJ^2X}e^{iJ^3X}) \tag{4.4}$$

We can extract from this the trace of the product of two and three symmetric polynomials as:

$$\begin{aligned}
&\frac{1}{N}Tr(X^{(i_1}\ldots X^{i_{l_1})}X^{(j_1}\ldots X^{j_{l_2})}) = \partial_{J^1_{i_1}}\ldots\partial_{J^1_{i_{l_1}}}\partial_{J^2_{j_1}}\ldots\partial_{J^2_{j_{l_2}}} I(J^1, J^2, 0)|_{J^{1,2}=0},\\
&\frac{1}{N}Tr(X^{(i_1}\ldots X^{i_{l_1})}X^{(j_1}\ldots X^{j_{l_2})}X^{(k_1}\ldots X^{k_{l_3})}) =\\
&\quad \partial_{J^1_{i_1}}\ldots\partial_{J^1_{i_{l_1}}}\partial_{J^2_{j_1}}\ldots\partial_{J^2_{j_{l_2}}}\partial_{J^3_{k_1}}\ldots\partial_{J^3_{k_{l_3}}} I(J^1, J^2, J^3)|_{J^{123}=0},
\end{aligned} \tag{4.5}$$

where $(\ldots)$ means the symmetrized product of X's.

The evaluation of $I(J^1, J^2, J^3)$ can be done if we note that the RHS of (4.4) is the trace of a product of three $SO(3)$ rotations with parameters $J^{1,2,3}$. The product of three rotations is itself a rotation with a parameter $J = J(J_1, J_2, J_3)$:

$$e^{iJX} = e^{iJ^1X}e^{iJ^2X}e^{iJ^3X}, \tag{4.6}$$

and the trace of this operator can be evaluated easily in a basis where JX is diagonal as:

$$I(J) \equiv I(J^1, J^2, J^3) = \frac{1}{N}\frac{sin(\frac{JN}{2})}{sin(\frac{J}{2})} \tag{4.7}$$

The dependence of J (or rather $cos(\frac{J}{2})$) on $J^{1,2,3}$ can be easy computed (see appendix 2) and we list here the result:

$$\begin{aligned}
cos(\frac{J}{2}) = {} & cos(\frac{J^1}{2})cos(\frac{J^2}{2})cos(\frac{J^3}{2}) - cos(\frac{J^1}{2})\frac{sin(\frac{J^2}{2})}{J^2}\frac{sin(\frac{J^3}{2})}{J^3} - cos(\frac{J^2}{2})\frac{sin(\frac{J^3}{2})}{J^3}\frac{sin(\frac{J^1}{2})}{J^1}\\
& - cos(\frac{J^3}{2})\frac{sin(\frac{J^1}{2})}{J^1}\frac{sin(\frac{J^2}{2})}{J^2} + \frac{sin(\frac{J^1}{2})}{J^1}\frac{sin(\frac{J^2}{2})}{J^2}\frac{sin(\frac{J^3}{2})}{J^3}(J^1 \times J^2)J^3
\end{aligned} \tag{4.8}$$

The cubic interaction in the case of fuzzy sphere introduces an additional subtlety, namely:

$$\frac{1}{N}Tr(Y^IY^JY^K) \neq \frac{1}{N}Tr(Y^IY^KY^J) \tag{4.9}$$

Because of this, some of the properties of cubic interaction we find in commutative case, are not there in the noncommutative case. In particular, we loose the appearance of the

Clebsch-Gordon coefficients in the cubic interaction. This asymmetry is not present in the case ϕ^3 interaction and not even in the case of $\phi_1^2\phi_2$, but it is present in the case $\phi_1\phi_2\phi_3$ type interaction, where $\phi_{1,2,3}$ are three different fields. We like to preserve those properties of interaction, as they seem to be present in the CFT [5], and we change the definition of the integration over the fuzzy sphere by replacing the trace over the product of harmonics with the trace over the symmetric product of harmonics. The change amounts in the end in dropping the last factor appearing in the expression of $cos(\frac{J}{2})$ (4.8), the only one not symmetric in $J^{1,2,3}$. After this change, we remain with J depending on the following variables only: $|J^1|$, $|J^2|$, $|J^3|$, J^1J^2, J^2J^3 and J^1J^3, where $|J| = \sqrt{J^2}$.

The method used for the evaluation of the two- and three- interaction for harmonics is given in (4.5). Spherical harmonics come with polynomial in X's that are both symmetric and traceless. The traceless property of polynomials is shifted to the traceless of partial derivatives in J^1, J^2 and J^3 (4.5) and as such, leads after setting J's to 0 to the following expression (we denote by $\tilde{A}^{(i)(j)}$ and $\tilde{A}^{(i)(j)(k)}$ the LHS of the equations (4.5) with the property that the indices i's, j's and k's are also traceless):

$$\begin{aligned}\tilde{A}^{(i)(j)} &= \delta_{l_1 l_2}\frac{1}{2^{2l_1}}(\frac{\partial}{\partial(cos(\frac{J}{2}))})^{l_1} I(J)|_{J=0}(\delta^{i_1 j_1}\ldots\delta^{i_{l_1} j_{l_1}}+\ldots),\\ \tilde{A}^{(i)(j)(k)} &= \frac{1}{2^{l_1+l_2+l_3}}(\frac{\partial}{\partial(cos(\frac{J}{2}))})^{\frac{l_1+l_2+l_3}{2}} I(J)|_{J=0}(\delta^{i_1 j_1}\ldots\delta^{i_{\alpha_1+1}k_1}\ldots+\ldots)\end{aligned} \tag{4.10}$$

The $\ldots$ in the last equations mean all possible contractions between indices i's and j's, j's and k's and i's and k's. We have also to specify:

$$(\frac{\partial}{\partial(cos(\frac{J}{2}))})^{l} I(J)|_{J=0} = \frac{(N+l)!}{N(N-1-l)!(2l+1)!!}, \tag{4.11}$$

and we give a derivation for this in the appendix. The final results for harmonics can be written in terms of those obtained in the commutative case (we denote the corresponding terms for the fuzzy sphere as $\langle Y^I Y^J\rangle_N$ and $\langle Y^I Y^J Y^K\rangle_N$):

$$\begin{aligned}\langle Y^I Y^J\rangle_N &= \frac{(N+l_1)!}{(2\rho)^{2l_1}N(N-l_1-1)!}\langle Y^I Y^J\rangle,\\ \langle Y^I Y^J Y^K\rangle_N &= \frac{(N+\frac{l_1+l_2+l_3}{2})!}{(2\rho)^{l_1+l_2+l_3}N(N-1-\frac{l_1+l_2+l_3}{2})!}\langle Y^I Y^J Y^K\rangle,\end{aligned} \tag{4.12}$$

where ρ is given (4.1). In the $N\to\infty$ limit, the factors coming from the fuzzy sphere go to 1 and the results for the commutative sphere are obtained.

5. Field theory on $AdS_2 \times S^2_{fuzzy}$

In this section we use the results obtained in the previous section and construct the correlation functions in the boundary theory arising from gravity on $AdS_2 \times S^2_{fuzzy}$.

We consider now the following action:

$$S = \int_{AdS_2} d^2x \sqrt{g} \frac{1}{N} Tr(\partial\phi\partial\phi + J_i\phi J_i\phi + \lambda\phi^3) \tag{5.1}$$

This action is the action (3.2) where the integration over sphere is replaced by the Tr over symmetric product of functions as discussed in the previous section. After the fuzzy sphere reduction, we obtain the same action on AdS_2 as in (5.1) but with A^{IJK} replaced by:

$$A_N^{IJK} = \frac{((N-1-l_1)!(N-1-l_2)!(N-1-l_3)!)^{\frac{1}{2}}}{(N-1-\frac{l_1+l_2+l_3}{2})!} \frac{N^{\frac{1}{2}}(N+\frac{l_1+l_2+l_3}{2})!}{((N+l_1)!(N+l_2)!(N+l_3)!)^{\frac{1}{2}}} A^{IJK} \tag{5.2}$$

After we reduce the theory on AdS_2 in the same way as before we obtain a similar result. Namely, the correlation functions in the case of fuzzy sphere are equal with those in the case of commutative sphere multiplied with the same factor as in (5.2). We give below the result for the chiral primaries correlation functions in this case (see (3.11)):

$$\begin{aligned} \langle O_I O_J O_K \rangle_N = &-\frac{\lambda C^{IJK}}{(2\pi)^{\frac{1}{2}}} \frac{((N-1-l_1)!(N-1-l_2)!(N-1-l_3)!)^{\frac{1}{2}}}{(N-1-\frac{l_1+l_2+l_3}{2})!} \times \\ &\times \frac{N^{\frac{1}{2}}(N+\frac{l_1+l_2+l_3}{2})!}{((N+l_1)!(N+l_2)!(N+l_3)!)^{\frac{1}{2}}} \frac{4\left((l_1^2-\frac{1}{4})(l_2^2-\frac{1}{4})(l_3^2-\frac{1}{4})\right)^{\frac{1}{2}}}{\alpha_1\alpha_2\alpha_3(\Sigma^2-1)}. \end{aligned} \tag{5.3}$$

Due to our simplified analysis, the equations above are not equal to those in CFT. Nevertheless, they contain all the essential ingredients of those. They have $SU(2)$ symmetry, that was represented as R-symmetry in CFT and more importantly, the results present a cutoff at $l_{1,2,3} = N$ and also at $\frac{l_1+l_2+l_3}{2} = N$. The overall factor that comes from the fuzzy sphere is of identical form the one coming from S_N permutations of the CFT. We find this as the clearest apperance of the non-comutativity in AdS/CFT. The characteristic features of the factorial terms identifying this non-commutativity are associated with the sphere and it is relevant to stress that the AdS contribution is not of this form. The full answer and especialy with various deformations of the AdS space itself contains further N dependence than just the overall factors that exibit agreement with the sphere. In addition, the gravitational coupling depends on N and there is are also $1/N$ loop effects that will give further contributions to the final N dependence. The full analysis of the complete N dependence on the gravity side is clearly non-trivial.

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6. Appendix 1

In this appendix we give the formulas and some derivations used in section 3. For the computations for harmonics we use the following expressions:

$$\begin{aligned} \frac{1}{4\pi}\int_{S^2}\frac{x^{i_1}\dots x^{i_l}}{\rho^l} &= \frac{\partial_{J_{i_1}}\dots\partial_{J_{i_l}}\int d^3x e^{-\frac{1}{2}x^2+Jx}|_{J=0}}{4\pi\int_0^\infty d\rho\rho^{l+2}e^{-\frac{1}{2}\rho^2}} = \\ &= \frac{\pi^{\frac{1}{2}}}{2^{\frac{l+2}{2}}\Gamma(\frac{l+3}{2})}(\delta^{i_1 i_2}\dots+\dots), \end{aligned} \tag{6.1}$$

where $\dots$ mean all possible contractions between i's. Then we obtain (3.4) from:

$$\langle Y^I Y^J\rangle = \Omega^I_{i_1\dots i_{l_1}}\Omega^J_{j_1\dots j_{l_1}}\frac{1}{4\pi}\int_S^2\frac{x^{i_1}\dots x^{i_l}x^{j_1}\dots x^{j_{l_2}}}{\rho^{l_1+l_2}} \tag{6.2}$$

and in a similar fashion (3.5). The extra combinatorial factors as $l!$ for (3.4) and $\alpha_1!\alpha_2!\alpha_3!$ and $\Sigma!$ come from different combinatorial way to match the indices for Ω's.

7. Appendix 2

In this appendix, we derive the formulas used in section 4. In order to derive the expression (4.8) we use the spin $\frac{1}{2}$ representation for $SO(3)$ rotations $e^{iJ^{1,2,3}x}$, replacing $x\to\frac{\sigma}{2}$ with σ being the Pauli matrices, and we obtain:

$$cos(\frac{J}{2})+i\frac{sin(\frac{J}{2})}{J}J\sigma = \prod_{k=1}^{3}(cos(\frac{J^k}{2})+i\frac{sin(\frac{J^k}{2})}{J^k}J^k\sigma) \tag{7.1}$$

From this, after straightforward algebraic manipulations we obtain (4.8).

For (4.11), we define:

$$K^l_{N+1}\equiv(\frac{d}{d(cosJ)})^l(\frac{sin(J(N+1))}{sinJ})|_{J=0} \tag{7.2}$$

and using $sin(J(N+1)) = cos(JN)sinJ + sin(JN)cosJ$ we obtain the following recurrence relations and conditions:

$$\begin{aligned} &K^l_{N+1} = K^l_N + (N+l)K^{l-1}_N, \quad l \geq 1, \\ &K^0_N = N, \\ &K^l_1 = 0, \quad l \geq 1, \end{aligned} \tag{7.3}$$

It is now easy to see that the expression in the RHS of (4.11) satisfies (7.3).

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